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LETTER TO THE EDITOR

A geometrical method for the stability analysis of dynamical systems

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Abstract. This work is an attempt to extend the geodesic equation on a manifold with an additional term of a linear mapping of the tangent space and to study the geometry of a class of curves defined by this extended version of the geodesic equation. The variational equation of a curve in this class is derived, and is used to examine stability of a trivial solution of a dynamical system whose configuration space is the diffeomorphism group of the circle. In particular, in regard to the KdV equation, this equation is compared with the Jacobi equation on the Bott–Virasoro group.

A solution of a dynamical system with a Lie–Poisson structure can be formulated as a geodesic in a Lie group with an appropriate metric [6]. In fact, this formulation has been applied to several systems (or equations), such as the Euler equations for a rigid body and for an inviscid incompressible fluid in a compact region [1], the periodic KdV equation [10] and the periodic filament equation [11]. The Lie groups associated with the systems just referred to are the rotation group SO_3, the diffeomorphism group $\mathcal{D}(M)$ of a compact Riemannian manifold M, the Bott–Virasoro group $\hat{\mathcal{D}}(S^1)$ and the loop group LSO_3, respectively. The metric on each Lie group is one-sided invariant and is taken to correspond to the Lagrangian of each system. A geodesic is a stationary curve of the length integral with respect to this metric and the length of its velocity vector is constant along it, which ensures Hamilton's principle.

In this formulation, an element of a Lie group represents a configuration of a system. Therefore the geometrical analysis of the Lie group enables us to study the behaviour of the system in the configuration space. In particular, solving the Jacobi equation, which governs infinitesimal variations of a geodesic, enables us to examine the stability of its solutions in the light of its configurations. It should be noted that the stability in this light differs from the conventional stability of solutions of the original evolution equation. In the case of inviscid incompressible fluids, for instance, it is known that there exist solutions stable in the latter sense but unstable in the former sense [7]. The aim of this work is to extend the stability analysis considered above to more general systems (possibly without Lie–Poisson structures). We introduce a class of curves in a manifold defined by an extension of the geodesic equation, and derive the variational equation of a curve in this class. This variational equation describes evolution of variation vector fields of the

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curve. As an example of the use of this equation, we consider a dynamical system whose configuration space is the diffeomorphism group of the circle.

We start by providing some definitions on the differential geometry which will be used later. The reader may wish to consult, e.g. [4]. Instead of a Lie group with a Riemannian metric (and so with a Riemannian connection), we will consider first a more general space, namely a manifold M with an affine connection ∇. The torsion and the curvature tensor fields on M, T and R, are defined as

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \tag{1}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{2}$$

where X, Y and Z are vector fields on M and $[\cdot,\cdot]$ denotes the bracket operation, $[X,Y]f = X(Yf) - Y(Xf)$. Let τ be a tensor field of type $(1,1)$ on M. For each $x \in M$, τ_x can be regarded as a linear mapping of the tangent space $T_x M$. This linear mapping will be denoted by $\tau(\cdot)$ (or $\tau_x(\cdot)$ at $x \in M$). Now we introduce the operator $\hat{\nabla}$ by writing

$$\hat{\nabla}_X Y = \nabla_X Y + \tau(X). \tag{3}$$

This operator defines a curve in M as an affine connection defines a geodesic; such a curve will be called a τ-geodesic. More precisely, a curve $\xi(t)$, $-\infty \leqq a < t < b \leqq \infty$, in M will be called a τ-geodesic if $\hat{\nabla}_{\dot{\xi}} \dot{\xi}$ exists and equals 0 for all t, where $\dot{\xi}$ denotes the velocity vector of ξ, i.e. $\dot{\xi} = \frac{\mathrm{d}}{\mathrm{d}t}\xi \in T_\xi M$. Before we derive the variational equation of a τ-geodesic, it is convenient first to examine the operator T_τ defined by

$$T_\tau(X,Y) = \nabla_X \tau(Y) - \nabla_Y \tau(X) - \tau([X,Y]). \tag{4}$$

It follows from this definition that

$$T_\tau(X,Y) = -T_\tau(Y,X) \tag{5}$$

$$T_\tau(fX,Y) = fT_\tau(X,Y) = T_\tau(X,fY) \tag{6}$$

in other words, T_τ is an antisymmetric tensor field.

Let us now consider the variational equation of a τ-geodesic, i.e. the equation which governs variation vector fields of a τ-geodesic. A variation of a τ-geodesic $\xi(t)$, $0 \leqq t \leqq 1$, is a one-parameter family of τ-geodesics $\phi^s(t)$, $-\epsilon < s < \epsilon$, such that $\phi^0(t) = \xi(t)$. More precisely, it is a smooth mapping of $[0,1] \times (-\epsilon,\epsilon)$ into M, $(t,s) \to \phi(t,s)$, such that (i) for each fixed $s \in (-\epsilon,\epsilon)$, $\phi^s(t) = \phi(t,s)$ is a τ-geodesic; (ii) $\phi^0(t) = \xi(t)$ for $0 \leqq t \leqq 1$ (cf [5]). Here let us define vector fields on M, X and J, by $X = \mathrm{d}\phi(\frac{\partial}{\partial t})$ and $J = \mathrm{d}\phi(\frac{\partial}{\partial s})$. For each fixed s, X is the velocity vector field of $\phi^s(t)$ and J is a variation vector field along $\phi^s(t)$. It is clear that X satisfies the τ-geodesic equation

$$\hat{\nabla}_X X = 0. \tag{7}$$

It also follows from the definitions of X and J that

$$[J,X] = \mathrm{d}\phi\left(\left[\frac{\partial}{\partial s},\frac{\partial}{\partial t}\right]\right) = 0 \tag{8}$$

and so

$$T(J,X) = \nabla_J X - \nabla_X J. \tag{9}$$

Using these equations, we can confirm that

$$\nabla_X \nabla_X J + \nabla_X T(J,X) + \nabla_X \tau(J) + R(J,X)X + T_\tau(J,X) = 0 \tag{10}$$

which, for $s = 0$, becomes the required equation:

$$\nabla_{\dot{\xi}}\nabla_{\dot{\xi}}J + \nabla_{\dot{\xi}}T(J, \dot{\xi}) + \nabla_{\dot{\xi}}\tau(J) + R(J, \dot{\xi})\dot{\xi} + T_\tau(J, \dot{\xi}) = 0. \tag{11}$$

This is a second-order linear differential equation for J. Hence J is uniquely determined by values of J and $\nabla_X J$ at one point of $\xi(t)$. Equation (11) and J will be called the τ-Jacobi equation and a τ-Jacobi field, respectively. Now we consider the particular case where ∇ is the Riemannian connection compatible with a Riemannian metric $\langle\cdot,\cdot\rangle$. Let $\xi(t)$ be a τ-geodesic such that $\dot{\xi}(0) = U$ and $J(t)$ a τ-Jacobi field along $\xi(t)$ such that $J(0) = 0$ and $\nabla_{\dot{\xi}}J(0) = V$. Then, the τ-Jacobi equation (11) can be used to express the Taylor series for $\langle J, J\rangle$ as

$$\begin{aligned}\langle J, J\rangle = \langle V, V\rangle t^2 - \langle\tau(V), V\rangle t^3 + (\tfrac{1}{3}(\langle\tau^2(V), V\rangle - \langle R(V, U)U, V\rangle - \langle T_\tau(V, U), V\rangle \\ -2\langle(\nabla_U\tau)(V), V\rangle) + \tfrac{1}{4}\langle\tau(V), \tau(V)\rangle)t^4 + \mathrm{O}(t^5).\end{aligned} \tag{12}$$

We have obtained the formula of the τ-Jacobi equation. As an example of the use of this equation, we will next consider a dynamical system whose configuration space is $\mathcal{D}(S^1)$, the group of orientation preserving smooth diffeomorphisms of the circle S^1. The group multiplication in $\mathcal{D}(S^1)$ is the composition of diffeomorphisms. $\mathcal{D}(S^1)$ with the inverse limit topology is an inverse limit Hilbert (ILH) Lie group. Therefore, we will assume the smooth manifold structure and the smooth group operations on $\mathcal{D}(S^1)$. (For details, see [3, 9].) Let $\mathfrak{X}(S^1)$ be the Lie algebra of $\mathcal{D}(S^1)$, i.e. the algebra of smooth vector fields on S^1. Elements of $\mathfrak{X}(S^1)$ will be denoted as $U = u\frac{\partial}{\partial x}, V = v\frac{\partial}{\partial x}, \ldots$, where $u, v, \ldots$ are functions on S^1. The commutator in $\mathfrak{X}(S^1)$ is the negative of the bracket operation for vector fields on S^1; that is, it is given by

$$[U, V] = -(uv_x - u_x v)\frac{\partial}{\partial x} \tag{13}$$

where we have introduced the short-hand notation for partial derivatives. Now let us suppose that $\mathfrak{X}(S^1)$ is equipped with the L^2 inner product

$$\langle U, V\rangle_e = \int_{S^1} uv\,\mathrm{d}x \tag{14}$$

and a linear mapping of the form

$$\tau_e(U) = \tau_e(u)\frac{\partial}{\partial x} = (a_1u_x + a_2u_{xx} + \cdots + a_nu_{nx})\frac{\partial}{\partial x} \tag{15}$$

where n is a positive integer. The inner product (14) and the linear mapping (15) induce on $\mathcal{D}(S^1)$ the right-invariant metric and the right-invariant tensor field of type $(1, 1)$, respectively. Let $\xi(t)$ be a τ-geodesic in $\mathcal{D}(S^1)$ with these metric and tensor field, and let $U(t)$ be the curve in $\mathfrak{X}(S^1)$ defined by $U = \mathrm{d}_\xi R_{\xi^{-1}}(\dot{\xi})$. Then, the τ-geodesic equation on $\mathcal{D}(S^1)$ is equivalent to the following evolution equation:

$$u_t + 3uu_x + \tau_e(u) = 0. \tag{16}$$

To show this, we investigate first the general case. Let G be a Lie group and $\mathcal{G}$ its Lie algebra with an inner product $\langle\cdot,\cdot\rangle_e$ and a linear mapping $\tau_e(\cdot)$. This inner product is extended by the right translation to induce the right-invariant metric on G; in fact, the metric on G, $\langle\cdot,\cdot\rangle$, is given by

$$\langle U_\xi, V_\xi\rangle = \langle\mathrm{d}_\xi R_{\xi^{-1}}(U_\xi), \mathrm{d}_\xi R_{\xi^{-1}}(V_\xi)\rangle_e \tag{17}$$

for $\xi \in G$ and $U_\xi, V_\xi \in T_\xi G$, where R_ξ denotes the right translation by ξ, i.e. $R_\xi(\eta) = \eta\xi$ for $\xi, \eta \in G$. The right invariance of this metric is obvious from the definition above.

The linear mapping τ_e is also extended by the right translation to induce the right-invariant tensor field of type $(1, 1)$ on G, $\tau(\cdot)$, which is given by

$$\tau(U_\xi) = \mathrm{d}_e R_\xi(\tau_e(\mathrm{d}_\xi R_{\xi^{-1}}(U_\xi))) \tag{18}$$

for $\xi \in G$ and $U_\xi \in T_\xi G$, where e is the identity element of G. Since the metric (17) and the tensor field (18) are right invariant, the τ-geodesic equation on G is reduced to an evolution equation on $\mathcal{G}$ in the following manner. Let $\xi(t)$ be a curve in G, $U(t)$ the curve in $\mathcal{G}$ defined by $U = \mathrm{d}_\xi R_{\xi^{-1}}(\dot{\xi})$ and ∇ the Riemannian connection compatible with the metric above. We begin with the formula

$$(\nabla_{\dot{\xi}}\eta)_\xi = \mathrm{d}_e R_\xi\left(\frac{\mathrm{d}}{\mathrm{d}t}V + (\nabla_{U_R}V_R)_e\right) \tag{19}$$

where η is a vector field along ξ and $V = \mathrm{d}_\xi R_{\xi^{-1}}(\eta)$ (see, e.g. [7]). It follows from this formula that, if $\xi(t)$ is a τ-geodesic, then the τ-geodesic equation is expressed in the form

$$(\hat{\nabla}_{\dot{\xi}}\dot{\xi})_\xi = \mathrm{d}_e R_\xi\left(\frac{\mathrm{d}}{\mathrm{d}t}U + (\nabla_{U_R}U_R)_e + \tau_e(U)\right) = 0 \tag{20}$$

which is satisfied if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}U + (\nabla_{U_R}U_R)_e + \tau_e(U) = 0. \tag{21}$$

This completes the demonstration that the τ-geodesic equation on G is reduced to an evolution equation on $\mathcal{G}$. Now we need to express the connection ∇ for right-invariant vector fields in terms of $\mathcal{G}$. For given $X \in \mathcal{G}$, let X_R denote the right-invariant vector field on G which at e takes the value X (i.e. $(X_R)_e = X$). Since the metric is right invariant, one obtains

$$2\langle\nabla_{U_R}V_R, W_R\rangle = \langle[U_R, V_R], W_R\rangle - \langle[U_R, W_R], V_R\rangle - \langle[V_R, W_R], U_R\rangle \tag{22}$$

for $U, V, W \in \mathcal{G}$. Thus the vector field $\nabla_{U_R}V_R$ at $\xi \in G$ is expressed as

$$(\nabla_{U_R}V_R)_\xi = -\mathrm{d}_e R_\xi(\tfrac{1}{2}([U, V] - \mathrm{ad}^*_U V - \mathrm{ad}^*_V U)) \tag{23}$$

where ad^* denotes the adjoint of ad with respect to the inner product $\langle ., .\rangle_e$, i.e.

$$\langle \mathrm{ad}^*_U V, W\rangle_e = \langle V, \mathrm{ad}_U W\rangle_e = \langle V, [U, W]\rangle_e \tag{24}$$

for $U, V, W \in \mathcal{G}$ (see, e.g. [2]). The minus sign enters the right-hand side of equation (23) because the Lie algebra is identified conventionally with the set of left- (not right-) invariant vector fields on the Lie group.

Now let us return to $\mathcal{D}(S^1)$, and obtain the τ-geodesic equation and the τ-Jacobi equation on it. The definition (24), with equations (13) and (14), gives $\mathrm{ad}^*_U V = (2u_x v + u v_x)\frac{\partial}{\partial x}$, and substitution into equation (23) yields

$$(\nabla_{U_R}V_R)_e = (2uv_x + u_x v)\frac{\partial}{\partial x}. \tag{25}$$

This and equation (21) at once give that the τ-geodesic equation on $\mathcal{D}(S^1)$ is equivalent to equation (16). Furthermore, let $\eta(t)$ be a τ-Jacobi field along the τ-geodesic $\xi(t)$ and $V = \mathrm{d}_\xi R_{\xi^{-1}}(\eta)$. Then, the τ-Jacobi equation is reduced to

$$v_{tt} + 2u_x v_t + 4uv_{tx} + 3u^2 v_{xx} + \tau_e(v_t) - \tau_e(u)v_x + \tau_e(u_x)v + \tau_e(uv_x) - \tau_e(u_x v) = 0. \tag{26}$$

This enables us to study the τ-geodesical stability of flows of the evolution equation (16). If the linear mapping τ_e is taken to be $\tau_e(U) = au_{xxx}\frac{\partial}{\partial x}$ so that (16) is the KdV equation, then the equation above becomes

$$v_{tt} + 2u_x v_t + 4uv_{tx} + 3u^2 v_{xx} + av_{txxx} - 3au_{xxx}v_x + 2au_x v_{xxx} + auv_{xxxx} = 0. \tag{27}$$

Since a solution of the KdV equation can be described as a geodesic in the Bott–Virasoro group $\hat{\mathcal{D}}(S^1)$ with the L^2 metric, it is of interest to compare equation (27) with the Jacobi equation on $\hat{\mathcal{D}}(S^1)$. The Bott–Virasoro group is the universal central extension of $\mathcal{D}(S^1)$. Its Lie algebra is the Virasoro algebra $\hat{\mathfrak{X}}(S^1)$, which is the universal central extension of $\mathfrak{X}(S^1)$. Elements of $\hat{\mathfrak{X}}(S^1)$ will be denoted as $\hat{U} = (u\frac{\partial}{\partial x}, a)$, $\hat{V} = (v\frac{\partial}{\partial x}, b), \ldots,$ where $a, b, \ldots \in R$ are central elements. The commutator in $\hat{\mathfrak{X}}(S^1)$ is given by $[\hat{U}, \hat{V}] = -((uv_x - u_x v)\frac{\partial}{\partial x}, \int_{S^1} u_{xx} v_x \mathrm{d}x)$. Let ∇ be the Riemannian connection on $\hat{\mathcal{D}}(S^1)$ compatible with the right-invariant metric which at the identity $\hat{e} \in \hat{\mathcal{D}}(S^1)$ is given by the L^2 inner product $\langle \hat{U}, \hat{V}\rangle_{\hat{e}} = \int_{S^1} uv\,\mathrm{d}x + ab$. Then, we obtain

$$(\nabla_{\hat{U}_R}\hat{V}_R)_{\hat{e}} = \left(2uv_x + u_x v + \tfrac{1}{2}av_{xxx} + \tfrac{1}{2}bu_{xxx}, \tfrac{1}{2}\int_{S^1} u_{xx}v_x\,\mathrm{d}x\right) \tag{28}$$

where $\hat{U}_R$ and $\hat{V}_R$ are right-invariant vector fields on $\hat{\mathcal{D}}(S^1)$ as before [8]. From this equation, it readily follows that, if $\hat{\xi}(t)$ is a geodesic in $\hat{\mathcal{D}}(S^1)$ and $\hat{U} = \mathrm{d}_{\hat{\xi}} R_{\hat{\xi}^{-1}}(\dot{\hat{\xi}})$, then the geodesic equation corresponds to the KdV equation $u_t + 3uu_x + au_{xxx} = 0$ with $a_t = 0$. Furthermore, it is straightforward to show that the Jacobi equation for a Jacobi field $\hat{\eta}(t)$ along a geodesic $\hat{\xi}(t)$ is equivalent to the following evolution equations for $\hat{V} = \mathrm{d}_{\hat{\xi}} R_{\hat{\xi}^{-1}}(\hat{\eta})$:

$$\begin{aligned} &v_{tt} + 2u_x v_t + 4uv_{tx} + 3u^2 v_{xx} + av_{txxx} - 3au_{xxx}v_x + 2au_x v_{xxx} + auv_{xxxx} \\ &\qquad + u_{xxx}\left(b_t + \int_{S^1} u_{xx}v_x\,\mathrm{d}x\right) = 0 \end{aligned} \tag{29a}$$

$$\left(b_t + \int_{S^1} u_{xx}v_x\,\mathrm{d}x\right)_t = 0. \tag{29b}$$

Since we are interested in perturbations of a KdV flow, we restrict $\hat{V}$ to a variation which does not vary a, the coefficient of the dispersive term in the KdV equation. In this case, b satisfies the relation

$$b_t + \int_{S^1} u_{xx}v_x\,\mathrm{d}x = a_s = 0 \tag{30}$$

so that equations (29a, b) are simplified to equation (27). That is, as far as perturbations of a KdV flow are concerned, the τ-Jacobi equation on $\mathcal{D}(S^1)$ is identical with the Jacobi equation on $\hat{\mathcal{D}}(S^1)$.

We now leave the KdV equation and return to the evolution equation (16) of the general form. This equation has the trivial solution $u(t, x) = c$, for which the τ-Jacobi equation (26) becomes

$$v_{tt} + 4cv_{tx} + \tau_e(v_t) + 3c^2 v_{xx} + c\tau_e(v_x) = 0. \tag{31}$$

Assuming the form $v(t, x) = \mathrm{e}^{\mathrm{i}(\omega t + 2\pi kx)}$ ($k \in \mathbb{Z}$), we obtain

$$(\omega + 2\pi ck)(\omega + 6\pi ck - \sum_{m=1}^{n} \mathrm{i}a_m(2\pi\mathrm{i}k)^m) = 0. \tag{32}$$

Thus, if $(-1)^m a_{2m} \geqq 0$ for all m, then the solution $u(t, x) = c$ is τ-geodesically stable in the sense that any τ-Jacobi field along the τ-geodesic corresponding to $u(t, x) = c$ is bounded for all time t. On the other hand, if $(-1)^m a_{2m} < 0$ for some m, then the solution $u(t, x) = c$ is τ-geodesically unstable in the sense that there exists an unbounded τ-Jacobi field along the τ-geodesic. In particular, if $\tau_e(U) = -\nu u_{xx}\frac{\partial}{\partial x}$ ($\nu > 0$) or $\tau_e(U) = au_{xxx}\frac{\partial}{\partial x}$ so that (16) is the Burgers equation or the KdV equation, then the solution $u(t, x) = c$

is τ-geodesically stable. It should be noted that the τ-geodesical stability of a fluid flow implies not the stability in the velocity field but that in the fluid (or particle) configuration.

In conclusion, we mention some future problems concerning the τ-Jacobi equation (11). Let τ_1 be a tensor field of type $(1, 1)$ on M and τ_0 a vector field on M. It is easy to generalize the τ-Jacobi equation for the case where a τ-geodesic $\xi(t)$ is defined by the equation of the more general form $\nabla_{\dot{\xi}}\dot{\xi} + \tau_1(\dot{\xi}) + \tau_0 = 0$. This generalization enables us to study the τ-geodesical stability of solutions of some systems with time-independent external force, e.g. the systems described by the Navier–Stokes equation with a forcing term. On the other hand, it seems that, if we specialize ∇ or $\tau(\cdot)$, then we can obtain more detailed results on the geometry of τ-geodesics. For instance, by taking equation (12) into account, we can define the geometrical quantity for τ-geodesics which corresponds to the sectional curvature for geodesics. These problems will be the subject of future work.

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